Yukawa Institute Kyoto
YITP-08-85

Shape invariance in prepotential approach to exactly solvable models

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(Dated: Jan 9, 2009)

Abstract

In supersymmetric quantum mechanics, exact-solvability of one-dimensional quantum systems can be classified only with an additional assumption of integrability, the so-called shape invariance condition. In this paper we show that in the prepotential approach we proposed previously, shape invariance is automatically satisfied and needs not be assumed.

PACS numbers: 03.65.Ca, 03.65.Ge, 02.30.Ik

Keywords: Prepotential, exact solvability, shape invariance, supersymmetry

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I. INTRODUCTION

It is generally known that exactly solvable systems are very rare in any branch of physics. Thus any new method to construct exactly solvable models would be of interest to the community concerned. It is therefore very interesting to realize that most exactly solvable one-dimensional quantum systems can be obtained in the framework of supersymmetric quantum mechanics (SUSYQM) [1, 2]. However, in SUSYQM, exact-solvability can be classified only with an additional assumption of integrability, so called shape invariance (SI) condition [3]. Hence in SUSYQM the SI condition must be taken as a sufficient condition for integrability at the outset. What is more, the transformation of the original coordinate, say x, to a new one $z = z(x)$ needed in solving the SI condition is not naturally determined within the framework of SUSYQM in most cases, but have to be taken as given from the known solutions of the respective models. It would be more satisfactory if the exact-solvability of a quantal system, including the required change of coordinates, could be determined with the simplest, and the most natural requirements.

In [4, 5, 6] a unified approach to both the exactly and quasi-exactly solvable systems is presented. This is a simple constructive approach, based on the so-called prepotential [7, 8, 9, 10, 11, 12, 13, 14, 15], which gives the potential as well as the eigenfunctions and eigenvalues simultaneously. The novel feature of the approach is that both exact and quasi-exact solvabilities can be solely classified by two integers, the degrees of two polynomials which determine the change of variable and the zero-th order prepotential. Hence this approach treats both quasi-exact and exact solvabilities on the same footing, and it provides a simple way to determine the required change of coordinates $z(x)$. All the well-known exactly solvable models given in [1, 2], most quasi-exactly solvable models discussed in [16, 17, 18, 19, 20], and some new quasi-exactly solvable ones (also for non-Hermitian Hamiltonians), can be generated by appropriately choosing the two polynomials.

Since all the well-known one-dimensional exactly solvable models obtained in SUSYQM, by taking SI condition as a sufficient condition, can also be derived without the SI condition in the prepotential approach, one wonders what role the SI condition plays in the latter approach. In this paper we would like to show that the SI condition is only a necessary condition in the prepotential approach to exactly solvable systems. Therefore, unlike SUSUQM, shape invariance needs not be assumed in the prepotential approach..

This paper is organized as follows. In Sect. II we give a brief review of the prepotential approach to exactly solvable models with both sinusoidal and non-sinusoidal coordinates. The idea of SI as a sufficient condition of integrability in SUSYQM is sketched in Sect. III. Sect. IV and V then demonstrate that in the prepotential approach for models with sinusoidal and non-sinusoidal coordinates, SI is automatically satisfied and needs not be imposed. Sect. VI concludes the paper.

II. PREPOTENTIAL APPROACH

The main ideas of the prepotential approach can be summarized as follows (we adopt the unit system in which $\hbar$ and the mass m of the particle are such that $\hbar = 2m = 1$). Consider a wave function $\phi_N(x)$ (N: non-negative integer) which is defined as

$$\phi_N(x) \equiv e^{-W_0(x)} p_N(z), \tag{1}$$

with

$$p_N(z) \equiv \begin{cases} 1, & N = 0; \\ \prod_{k=1}^{N} (z - z_k), & N > 0. \end{cases} \tag{2}$$

Here $z = z(x)$ is some real function of the basic variable x, $W_0(x)$ is a regular function of $z(x)$, and z_k's are the roots of $p_N(z)$. The variable x is defined on the full line, half-line, or finite interval, as dictated by the choice of $z(x)$. The function $p_N(z)$ is a polynomial in an $(N+1)$-dimensional Hilbert space with the basis $\langle 1, z, z^2, \ldots, z^N \rangle$. $W_0(x)$ defines the ground state wave function.

The wave function ϕ_N can be recast as

$$\phi_N = \exp\left(-W_N(x, \{z_k\})\right), \tag{3}$$

with W_N given by

$$W_N(x, \{z_k\}) = W_0(x) - \sum_{k=1}^{N} \ln|z(x) - z_k|. \tag{4}$$

Operating on ϕ_N by the operator $-d^2/dx^2$ results in a Schrödinger equation $H_N \phi_N = 0$, where

$$H_N = -\frac{d^2}{dx^2} + V_N, \tag{5}$$

$$V_N \equiv W_N'^2 - W_N''. \tag{6}$$

Here prime represents differentiation with respect to x. It is seen that the potential V_N is defined by W_N, and we shall call W_N the Nth order prepotential. From Eq. (4), one finds that V_N has the form $V_N = V_0 + \Delta V_N$:

$$\begin{aligned} V_0 &= W_0'^2 - W_0'', \\ \Delta V_N &= -2\left(W_0' z' - \frac{z''}{2}\right)\sum_{k=1}^{N}\frac{1}{z-z_k} + \sum_{\substack{k,l\\ k\neq l}}\frac{z'^2}{(z-z_k)(z-z_l)}. \end{aligned} \tag{7}$$

Thus the form of V_N, and consequently its solvability, are determined by the choice of $W_0(x)$ and z'^2 (or equivalently by $z'' = (dz'^2/dz)/2$). Let $W_0'z' = P_m(z)$ and $z'^2 = Q_n(z)$ be polynomials of degree m and n in z, respectively. In [4], it was shown that if the degree of $W_0'z'$ is no higher than one ($m \leq 1$), and the degree of z'^2 no higher than two ($n \leq 2$), then in $V_N(x)$ the parameter N and the roots z_k's, which satisfy the so-called Bethe ansatz equations (BAE) to make the potential analytic, will only appear in an additive constant and not in any term involving powers of z. Such system is then exactly solvable. If the degree of one of the two polynomials exceeds the corresponding upper limit, the resulted system is quasi-exactly solvable. The transformed coordinates $z(x)$ such that the degree of z'^2 is no higher than two are called sinusoidal coordinates. There are six types of one-dimensional exactly solvable models which are based on such coordinates, namely, the shifted-oscillator, three-dimensional oscillator, Morse, Scarf type I and II, and generalized Pöschl-Teller models as list in [1].

In [6], the prepotential approach to exactly solvable systems was extended to systems based on non-sinusoidal transformed variable $z(x)$ which is a solution of $z' = \lambda - z^2$. With this, the remaining four types of exactly solvable systems listed in [1], namely, the Coulomb, Eckart, and Rosen-Morse type I and II models, are also covered by the prepotential approach.

A. Sinusoidal coordinates

For exactly solvable models with sinusoidal coordinates we take $m = 1$ and $n = 2$, i.e., $P_1(z) = az + b$, and $Q_2(z) = \alpha z^2 + \beta z + \gamma$, where $a,\ b,\ \alpha,\ \beta$ and γ are real constants. With these choices we obtain [4]

$$V_N = W_0'^2 - W_0'' + \alpha N^2 - 2aN - 2\sum_{k=1}^{N}\frac{1}{z-z_k}\left\{\left(a - \frac{\alpha}{2}\right)z_k + b - \frac{\beta}{4} - \sum_{l\neq k}\frac{Q_2(z_k)}{z_k - z_l}\right\}. \tag{8}$$

Demanding the residues at z_k's vanish gives the set of Bethe ansatz equations

$$\left(a - \frac{\alpha}{2}\right) z_k + b - \frac{\beta}{4} - \sum_{l \neq k} \frac{Q_2(z_k)}{z_k - z_l} = 0, \quad k = 1, 2, \ldots, N. \tag{9}$$

With this set of roots z_k, the last term in Eq. (8) vanishes, and we obtain a potential $V_N(x) = V_0(x) - E_N$ without simple poles. Here $V_0(x) = W_0'^2 - W_0''$ does not involve N and z_k's, and can be taken as the exactly solvable potential of the system with eigen-energies $E_N = 2aN - \alpha N^2$. In fact, $V_0(x)$ is exactly the supersymmetric form presented in [1] for the shifted-oscillator, three-dimensional oscillator, Morse, Scarf type I and II, and generalized Pöschl-Teller models (for easy comparison, we note that α and a here equal $\pm\alpha^2$ and αA in [1]).

B. Non-sinusoidal coordinates

As mentioned before, the Coulomb, Eckart, and Rosen-Morse type I and II models involve a change of coordinates of the form $z' = \lambda - z^2$ which is non-sinusoidal. But with a slight extension of the methods in [4], all these four models can be treated in a unified way in the prepotential approach [6]. The extension is simply to allow the coefficients in W_0 be dependent on N. It turns out that W_0' takes the form

$$W_0'(N) = -\left(A + N\alpha\right) z + \frac{B}{A + N\alpha}, \tag{10}$$

where A and B are real parameters. Then the potential V_N becomes $V_N(x) = V(x) - E_N$, where

$$V(x) = A\left(A - 1\right) z^2(x) - 2Bz(x), \tag{11}$$

and

$$E_N = -\frac{B^2}{\left(A + N\right)^2} - \lambda \left[A\left(2N + 1\right) + N^2\right]. \tag{12}$$

Now $V(x)$ is independent of N, and can be taken to be the potential of an exactly solvable system, with eigenvalues E_N $(N = 0, 1, 2, \ldots)$. The corresponding wave functions ϕ_N are given by (1):

$$\phi_N \sim e^{(A+N)\int^x dx z(x) - \frac{B}{A+N} x}\, p_N(x), \quad N = 0, 1, \ldots \tag{13}$$

The BAE satisfied by the roots z_k's are

$$\sum_{l\neq k}\frac{z_k^2-\lambda}{z_k-z_l}-(A+N-1)\,z_k+\frac{B}{A+N}=0,\quad k=1,2,\ldots,N. \tag{14}$$

Finally,we mention here that $V(x)$ in (11) can be obtained, up to an additive constant, from $W_0(N)$ with any value of N. Particularly, the form adopted in supersymmetric quantum mechanics (e.g., in [1]) is obtained from the zero-th order prepotential $W_0(N=0)$ with $N=0$ [6].

III. SHAPE INVARIANCE IN SUPERSYMMETRIC QUANTUM MECHANICS

From the discussions in the last section, we see that in the prepotential approach, exactly solvable models are determined by the zero-th order prepotential $W_0(x)$ in the sinusoidal cases, or $W_0 \equiv W_0(N=0)$ with $N=0$ in the four non-sinusoidal cases. The potential V_0 is completely determined by W_0: $V_0 = W_0'^2 - W_0''$, and consequently, the Hamiltonian $H_0 = -d^2/dx^2 + V_0$ is factorizable as $H_0 = A^+A$ with the first-order operators

$$A \equiv \frac{d}{dx}+W_0', \qquad A^+ \equiv -\frac{d}{dx}+W_0'. \tag{15}$$

This fact is indeed the base of SUSYQM. In SUSYQM [1, 2] one considers the relation between the spectrum of H_0 and that of its so-called super-partner Hamiltonian H_1 constructed according to $H_1 \equiv AA^+ = -d^2/dx^2 + V_1$, where $V_1 \equiv W_0'^2 + W_0''$. In forming V_1, it is equivalent to using a prepotential $-W_0$. The ground state of H_1 is therefore $\exp(W_0)$, and it follows that the ground states of H_0 and H_1 cannot be both normalizable.

Let us suppose the ground state of H_0, i.e. $\exp(-W_0)$, is normalizable, and denote the normalized eigenfunctions of the Hamiltonians $H_{0,1}$ by $\psi_n^{(0,1)}$ with eigenvalues $E_n^{(0,1)}$, respectively. Here the subscript $n=0,1,2,\ldots$ denotes the number of nodes of the wave function. It is easily proved that V_0 and V_1 have the same energy spectrum except for the ground state of V_0 with $E_0^{(0)}=0$, which has no corresponding level for V_1 [1, 2]. More explicitly, we have the following supersymmetric relations:

$$\begin{gathered} E_n^{(1)} = E_{n+1}^{(0)}, \\ \psi_n^{(1)} = \left(E_{n+1}^{(0)}\right)^{-1/2} A\psi_{n+1}^{(0)}, \quad A\psi_0^{(0)} = 0, \\ \psi_{n+1}^{(0)} = \left(E_n^{(1)}\right)^{-1/2} A^+\psi_n^{(1)}. \end{gathered} \tag{16}$$

Hence A annihilates $\psi_0^{(0)}$, and converts an eigenfunction of an excited state of H_0 into an eigenfunction of H_1 with the same energy, but with one less number of nodes, while A^+ does the reverse. Consequently, if the spectrum of one system is exactly known, so is the spectrum of the other.

This is, however, all that supersymmetry says about the two partner potentials. If any one of the spectra is unknown, then supersymmetry is useless in solving them. It is therefore gratifying that most of the well-known one-dimensional exactly solvable models process a property called shape invariance. With hindsight, one can then impose shape invariance as an additional requirement along with supersymmetry to classify exactly solvable systems having such property. This has been done and most exactly solvable systems are then unified within the framework of SUSYQM [1, 2].

Shape invariance means that the two super-partner potentials V_0 and V_1 are related by the relation

$$V_1(x;\boldsymbol{\lambda}_0) = V_0(x;\boldsymbol{\lambda}_1) + R(\boldsymbol{\lambda}_0), \tag{17}$$

where $\boldsymbol{\lambda}_0$ is a set of parameters of the original V_0, $\boldsymbol{\lambda}_1 = f(\boldsymbol{\lambda}_0)$ is a function of $\boldsymbol{\lambda}_0$, and $R(\boldsymbol{\lambda}_0)$ is a constant which depends only $\boldsymbol{\lambda}_0$. This implies

$$W_0^{\prime 2}(x,\boldsymbol{\lambda}_0) + W_0^{\prime\prime}(x,\boldsymbol{\lambda}_0) = W_0^{\prime 2}(x,\boldsymbol{\lambda}_1) - W_0^{\prime\prime}(x,\boldsymbol{\lambda}_1) + R(\boldsymbol{\lambda}_0). \tag{18}$$

Eq. (17) implies that V_1 has the same shape as that of V_0, but is defined by parameters $\boldsymbol{\lambda}_1$ instead of $\boldsymbol{\lambda}_0$. From (18) one deduces that the ground state wave function of V_1 is $\psi_0^{(1)} \sim \exp(-W_0(x,\boldsymbol{\lambda}_1)$ with energy $R_0(\boldsymbol{\lambda}_0)$. Then from (16) we know the energy of the first excited state of V_0 to be $R(\boldsymbol{\lambda}_0)$, and the wave function $\psi_1^{(0)} \sim A^+\psi_0^{(1)}$. By repeated use of the shape invariance condition, one can construct the partner V_2 of V_1, V_3 of V_2, etc. The ground state wave function of V_n ($n = 0, 1, \ldots$) is $\psi_0^{(n)} \sim \exp(-W_0(x,\boldsymbol{\lambda}_n)$, where $\boldsymbol{\lambda}_n = f^n(\boldsymbol{\lambda}_0)$, with energy $\sum_{k=0}^{n-1} R(\boldsymbol{\lambda}_k)$. Then again from (16) we know that the wave function of the n^{th} state of H_0 is $\psi_n^{(0)} \sim (A^+)^n\psi_0^{(n)}$, with energy

$$E_n^{(0)} = \sum_{k=0}^{n-1} R(\boldsymbol{\lambda}_k), \quad n = 0, 1, \ldots \tag{19}$$

So with shape invariance one obtains the complete spectrum of H_0.

It is now obvious that SI is a sufficient condition of integrability in SUSYQM. To classify shape-invariant exactly solvable models in SUSYQM, one must solve the SI condition (18)

to get all the functional forms of $W_0(x)$, $\boldsymbol{\lambda}_1 = f(\boldsymbol{\lambda}_0)$, and $R(\boldsymbol{\lambda}_0)$. This general problem is very difficult and, to the best of our knowledge, is still unsolved. Further constraints on the possible class of shape invariant potentials are required. Particularly, in order to obtain the well-known exactly solvable models one must assume that (again with hindsight) the parameters of the two partner potentials are related by simply a translational shift, i.e. $\boldsymbol{\lambda}_1 = f(\boldsymbol{\lambda}_0) = \boldsymbol{\lambda}_0 + \mathbf{m}$ differ from $\boldsymbol{\lambda}_0$ only by a set of constants $\mathbf{m}$. Even with this simplification, the required change of coordinates $z = z(x)$ needed in solving the SI condition cannot be determined naturally in the approach of SUSYQM, but has to be taken as given from the known solutions of the respective models.

On the other hand, in the prepotential approach SI needs not be imposed, and W_0 and $z(x)$ are determined by simply picking two polynomials with the appropriate degrees. In this sense it appears to us that the prepotential approach is conceptually much simpler. Nevertheless, putting the differences of the two approaches aside, one could not help but wonder what role SI plays in the prepotential approach. Below we would like to demonstrate that for the exactly solvable models obtained in the prepotential approach, SI is automatically satisfied. We shall discuss the cases with sinusoidal and non-sinusoidal coordinates separately.

IV. SHAPE INVARIANCE IN PREPOTENTIAL APPROACH: SINUSOIDAL COORDINATES

Our strategy is to show that, with $z(x)$ and $W_0(x)$ given in Sect. II(A) and (B) that produce the ten well-known exactly solvable models, the SI condition (18) is always satisfied, i.e. one can always find the set of new parameters $\boldsymbol{\lambda}_1$ in terms of the old ones $\boldsymbol{\lambda}_0$. In the process, we demonstrate that the change in the parameters of the shape-invariant potentials are translational.

In this section, we first consider the cases involving sinusoidal coordinates. For exactly solvable systems, we must take $W_0'z' = P_1(z)$. Labelling the corresponding parameters of

the two shape-invariant potentials by $k = 0, 1$, we have

$$z'^2 = Q_2(z) = \alpha z^2 + \beta z + \gamma; \tag{20}$$

$$P_1^{(k)}(z) = a_k z + b_k, , \quad k = 0, 1 \tag{21}$$

$$W_0'(\boldsymbol{\lambda}_k) = \frac{P_1^{(k)}(z)}{\sqrt{Q_2(z)}}, \quad \boldsymbol{\lambda}_k = (a_k, b_k). \tag{22}$$

Note that $z(x)$ is the same for the shape-invariant potentials. Then the SI condition (18) leads to

$$\left(P_1^{(0)2} - P_1^{(1)2}\right) + Q_2 \frac{d}{dz}\left(P_1^{(0)} + P_1^{(1)}\right) - \frac{1}{2}\frac{dQ_2}{dz}\left(P_1^{(0)} + P_1^{(1)}\right) = R(\boldsymbol{\lambda}_0) Q_2. \tag{23}$$

Equating the coefficients of the powers of z, one arrives at the following equations relating the parameters

$$\begin{gathered} a_0^2 - a_1^2 = R\alpha, \\ 2\left(a_0 b_0 - a_1 b_1\right) + \frac{\beta}{2}\left(a_0 + a_1\right) - \alpha\left(b_0 + b_1\right) = R\beta, \\ b_0^2 - b_1^2 + \gamma\left(a_0 + a_1\right) - \frac{\beta}{2}\left(b_0 + b_1\right) = R\gamma. \end{gathered} \tag{24}$$

For simplicity we write R for $R(\boldsymbol{\lambda}_0)$. We mention here that the signs of a and b are fixed by the normalization of the wave functions. This means they are the same for the two shape-invariant partner potentials.

We would like to solve (24) for $\boldsymbol{\lambda}_1 = (a_1, b_1)$ and R in terms of $\boldsymbol{\lambda}_0 = (a_0, b_0)$. To facilitate solution, we find it convenient to first determine all inequivalent types of sinusoidal coordinates.

A. Inequivalent sinusoidal coordinates

Depending on the presence of the parameters α, β and γ, there are three inequivalent cases of sinusoidal coordinates: (i) $z'^2 = \gamma \neq 0$, (ii) $z'^2 = \beta z + \gamma$ ($\beta \neq 0$), and (iii) $z'^2 = \alpha z^2 + \beta z + \gamma$ ($\alpha \neq 0$). By an appropriate shifting and/or scaling, these cases can be recast into three canonical forms.

The form given for case (i) is already the canonical form of this case. We shall take $\gamma > 0$ as $\gamma \leq 0$ leads to physically uninteresting change of variable. This case gives rise to the shifted oscillator.

By shifting z to $\hat{z} \equiv z + \gamma/\beta$ in case (ii), we get the canonical form $\hat{z}'^2 = \beta\hat{z}$. For physical systems we require $\beta > 0$. This case corresponds to the three-dimensional oscillator.

Case (iii) can be recast as $\tilde{z}'^2 = \alpha\tilde{z}^2 + \tilde{\gamma}$, where $\tilde{z} \equiv z + \beta/2\alpha$ and $\tilde{\gamma} \equiv \Delta/4\alpha$ with the discriminant $\Delta \equiv 4\alpha\gamma - \beta^2$. For the case $\Delta = 0$ (the exponential case) and $\alpha > 0$, the system thus generated is related to the Morse potential. For $\Delta \neq 0$, we have two situations. If $\alpha > 0$ (the hyperbolic case), the canonical form is $\hat{z}'^2 = \alpha(\hat{z}^2 \pm 1)$, where $\hat{z} \equiv \sqrt{4\alpha^2/|\Delta|}\tilde{z}$, and the plus (minus) sign corresponds to $\Delta > 0$ ($\Delta < 0$). The plus sign gives rise to the Scarf II model, while the minus sign corresponds to the generalized Pöschl-Teller model. For $\alpha < 0$ (the trigonometric case), the canonical form is $\hat{z}'^2 = |\alpha|(\pm 1 - \hat{z}^2)$, where again $\hat{z} \equiv \sqrt{4\alpha^2/|\Delta|}\tilde{z}$, and the plus (minus) sign corresponding to $\Delta < 0$ ($\Delta > 0$). With the plus sign we get the Scarf I model, while the minus sign does not lead to any viable system as the transformation is imaginary.

From the above discussions, we see that we need only to discuss the three inequivalent canonical cases, namely, (i) $z'^2 = \gamma \neq 0$, (ii) $z'^2 = \beta z$ ($\beta > 0$), and (iii) $z'^2 = \alpha(z^2 + \delta)$ ($\delta = 0, \pm 1$ for $\alpha > 0$, and $\delta = -1$ if $\alpha < 0$).

B. Case (i): $z'^2 = \gamma > 0$

For this case, it is easy to check that a_0 (a_1) must not vanish, or it will lead to vanishing potential. Furthermore, we must have $a_0 > 0$ and $a_1 > 0$ in order that the wave functions be normalizable. The SI conditions (24) become

$$(a_0 + a_1)(a_0 - a_1) = 0, \tag{25}$$

$$a_0 b_0 - a_1 b_1 = 0, \tag{26}$$

$$b_0^2 - b_1^2 + \gamma\,(a_0 + a_1) = R\gamma. \tag{27}$$

Equations (25) and (26) require $a_1 = a_0$, $b_1 = b_0$, or $a_1 = -a_0$, $b_1 = -b_0$. In the latter solution the signs of a_1 and b_1 are different from those of a_0 and b_0, and hence the wave functions of one of the two systems cannot be normalizable if those of the other system can. In fact, for this case we have $R = 0$ from (27). This means the ground states of the two systems have the same energy. But the flip of both signs of a and b of W_0 means that the ground states of the two systems have the forms $\exp(-W_o)$ and $\exp(+W_0)$. They cannot be both normalizable. This is exactly the result in SUSYQM.

So we are left with the choice $a_1 = a_0,\ b_1 = b_0$. From (27) we have $R = 2a_0$. Thus R is a constant, and from (19) it implies oscillator-like spectrum, i.e. $E_n = na_0$. This gives the shifted oscillator.

The above discussion shows that in this case SI is a necessary condition. The parameters of the two partner systems are related by $(a_1,\ b_1) = (a_0,\ b_0)$, and the shift parameter is $R = 2a_0$.

C. Case (ii): $z'^2 = \beta z$ ($\beta > 0$)

Normalizability of wave functions in this case require that $a > 0$ and $b < 0$. Now the SI conditions (24) are

$$(a_0 + a_1)(a_0 - a_1) = 0, \tag{28}$$

$$2\left(a_0 b_0 - a_1 b_1\right) + \frac{\beta}{2}\left(a_0 + a_1\right) = R\beta, \tag{29}$$

$$\left(b_0 + b_1\right)\left(b_0 - b_1 - \frac{\beta}{2}\right) = 0. \tag{30}$$

Possible solutions of these equations are $a_0 \pm a_1 = 0$, $b_0 + b_1 = 0$ or $b_0 - b_1 - \beta/2 = 0$. To keep the signs of a and b unchanged, we can only take $(a_1,\ b_1) = (a_0,\ b_0 - \beta/2)$ as the viable solution. Then from (30) we get $R = 2a_0$, which again gives an oscillator-like spectrum. This is just the case of the three-dimensional oscillator.

D. Case (iii): $z'^2 = \alpha(z^2 + \delta)$

Next we consider the case with $z'^2 = \alpha(z^2 + \delta)$ ($\delta = 0, \pm 1$ for $\alpha > 0$, and $\delta = -1$ if $\alpha < 0$). As mentioned before, this case covers the Morse, generalized Pöschl-Teller, and the Scarf I and II potentials. The SI conditions (24) are

$$a_0^2 - a_1^2 = R\alpha, \tag{31}$$

$$2\left(a_0 b_0 - a_1 b_1\right) - \alpha\left(b_0 + b_1\right) = 0, \tag{32}$$

$$b_0^2 - b_1^2 + \alpha\delta\left(a_0 + a_1\right) = R\alpha\delta. \tag{33}$$

To solve a_1 , b_1 and R in terms of a_0 and b_0, we eliminate $R\alpha$ in (33) using (31) to get

$$\left(b_0 + b_1\right)\left(b_0 - b_1\right) + \delta\left(a_0 + a_1\right)\left(a_1 - a_0 + \alpha\right) = 0. \tag{34}$$

From (34) we can have four possible sets of solutions:

$$a_0 + a_1 = 0, \quad b_0 + b_1 = 0; \tag{35}$$

$$a_0 + a_1 = 0, \quad b_0 - b_1 = 0; \tag{36}$$

$$a_0 - a_1 = \alpha, \quad b_0 + b_1 = 0; \tag{37}$$

$$a_0 - a_1 = \alpha, \quad b_0 - b_1 = 0. \tag{38}$$

The first three sets of solutions involve change of signs of a and/or b, and so are not viable as discussed before. Thus for this case we must take $(a_1,\ b_1) = (a_0 - \alpha,\ b_0)$ which also satisfies (32). Eq. (31) then gives

$$R(\boldsymbol{\lambda}_0) = \frac{a_0^2 - a_1^2}{\alpha} = 2a_0 - \alpha. \tag{39}$$

From (19) the energies are

$$\begin{aligned} E_n &= \frac{a_0^2 - a_n^2}{\alpha} \\ &= \frac{a_0^2 - (a_0 - n\alpha)^2}{\alpha}, \quad n = 0, 1, \ldots \end{aligned} \tag{40}$$

This is exactly the results in SUSYQM [1].

To conclude this section, we have shown that SI is automatically satisfied in the prepotential approach for the sinusoidal cases.

V. SHAPE INVARIANCE IN PREPOTENTIAL APPROACH: NON-SINUSOIDAL COORDINATES

In this case, $W_0' = -Az + B/A$ and $z' = \alpha(\lambda - z^2)$. Here $\boldsymbol{\lambda}_0 = (A, B)$. As in the last section, we show that one can always find a set of new parameter $\boldsymbol{\lambda}_1 = (A', B')$ in terms of $\boldsymbol{\lambda}_0$ that solves the SI condition (18). In fact, from (18) one finds

$$A\,(A + \alpha) = A'\,(A' - \alpha)\,, \tag{41}$$

$$B = B', \tag{42}$$

$$\frac{B^2}{A^2} - \alpha\lambda A = \frac{B'^2}{A'^2} + \alpha\lambda A' + R. \tag{43}$$

Solutions of (41) are $A' = -A$ and $A' = A + \alpha$. The first solution has the sign of A changed, and will lead to non-normalized wave functions. Hence the viable solution is

$\boldsymbol{\lambda}_1 = (A', B') = (A + \alpha, B)$. Once again, the change in the parameters A and B of the shape-invariant potentials are translational. Finally, from (43) we find

$$R(\boldsymbol{\lambda}_0) = B^2 \left[\frac{1}{A^2} - \frac{1}{(A+\alpha)^2} \right] - \alpha\lambda\,(2A + \alpha)\,. \tag{44}$$

This agrees with the results in SUSYQM [1].

Thus we have shown that in the prepotential approach for models based on non-sinusoidal coordinates, SI is also a necessary consequence of the forms of W_0 and z'.

VI. SUMMARY

A unified approach to both the exactly and quasi-exactly solvable systems has been proposed previously based on the so-called prepotential in [4, 5, 6]. In this approach solvability of a quantal system can be solely classified by two integers, the degrees of two polynomials which determine the change of variable and the zero-th order prepotential. All the well-known exactly solvable models obtained in SUSYQM can be easily constructed by appropriately choosing the two polynomials.

But all these exactly solvable models are obtained in SUSYQM only by taking the SI condition as a sufficient condition. The requirement to get exactly solvable models in the prepotential approach appears to be much simpler, and definitely without the need of SI condition. In this paper we have shown that the SI condition is in fact only a necessary condition in the prepotential approach to exactly solvable systems, and hence needs not be assumed. In the process, we have demonstrated that the change in the parameters of the well-known shape-invariant potentials are indeed translational, a result which was also assumed in SUSYQM.

Acknowledgments

This work is supported in part by the National Science Council (NSC) of the Republic of China under Grant Nos. NSC 96-2112-M-032-007-MY3 and NSC 95-2911-M-032-001-MY2. Part of the work was done during my visit to the Yukawa Institute for Theoretical Physics (YITP) at the Kyoto University supported under NSC Grant No. 97-2918-I-032-002. I would like to thank R. Sasaki and the staff and members of YITP for their hospitality. I

am also grateful to Y. Hosotani and Y. Matsuo for useful discussion and hospitality.

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